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# An Improvement of the Griffiths–Hurst–Sherman Inequality for the Ising Ferromagnet

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We prove the following inequality for the truncated correlation in the Ising model in zero external field:

$$\begin{aligned}&\langle\sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_l\rangle - \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_l\rangle - \langle\sigma_i\sigma_l\rangle\langle\sigma_j\sigma_k\rangle\\ &\quad\leqslant -2\langle\sigma_i\sigma_m\rangle\langle\sigma_j\sigma_m\rangle\langle\sigma_k\sigma_m\rangle\langle\sigma_l\sigma_m\rangle\\ &\qquad -2(\langle\sigma_i\sigma_k\rangle - \langle\sigma_i\sigma_m\rangle\langle\sigma_m\sigma_k\rangle)(\langle\sigma_j\sigma_k\rangle - \langle\sigma_j\sigma_m\rangle\langle\sigma_m\sigma_k\rangle)\langle\sigma_k\sigma_l\rangle\\ &\qquad -2\langle\sigma_i\sigma_m\rangle\langle\sigma_j\sigma_m\rangle(\langle\sigma_i\sigma_k\rangle - \langle\sigma_i\sigma_m\rangle\langle\sigma_m\sigma_k\rangle)(\langle\sigma_i\sigma_l\rangle - \langle\sigma_i\sigma_m\rangle\langle\sigma_m\sigma_l\rangle)\end{aligned}$$

This inequality is a strengthening of the Lebowitz inequality for the four-point function and implies the following improvement of the GHS inequality:

$$\langle\sigma_i;\sigma_j;\sigma_k\rangle^T \leqslant -2\langle\sigma_i;\sigma_k\rangle^T\langle\sigma_j;\sigma_k\rangle^T\langle\sigma_k\rangle$$

This in turn implies the critical exponent inequality

$$\Delta_3' \geqslant \gamma' - \beta$$

**KEY WORDS:** Correlation inequalities; Ising model.

## 1. INTRODUCTION

We consider an Ising model with spins $\sigma_i = \pm 1$ on sites $i = 1, \ldots, N$ and Hamiltonian

$$-H = \sum_{1 \leqslant i < j \leqslant N} J_{ij}\sigma_i\sigma_j \tag{1.1}$$

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with $J_{ij} \geqslant 0$. The partition function

$$Z = \frac{1}{2^N} \sum_{\sigma_i = \pm 1} e^{-\beta H}$$

where $\beta$ is the inverse temperature and expectations are defined by

$$\langle \cdot \rangle = \frac{1}{2^N} \sum_{\sigma_i = \pm 1} (\cdot) e^{-\beta H}/Z \tag{1.2}$$

Our main result is the following.

**Theorem 1.** Let $J_{ij} \geqslant 0$ in (1.1). Then for all $m$

$$\begin{aligned}
&\langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_j\rangle\langle \sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_k\rangle\langle \sigma_j\sigma_l\rangle - \langle \sigma_i\sigma_l\rangle\langle \sigma_j\sigma_k\rangle \\
&\quad \leqslant -2\langle \sigma_i\sigma_m\rangle\langle \sigma_j\sigma_m\rangle\langle \sigma_k\sigma_m\rangle\langle \sigma_l\sigma_m\rangle \\
&\qquad -2(\langle \sigma_i\sigma_k\rangle - \langle \sigma_i\sigma_m\rangle\langle \sigma_m\sigma_k\rangle)(\langle \sigma_j\sigma_k\rangle - \langle \sigma_j\sigma_m\rangle\langle \sigma_m\sigma_k\rangle)\langle \sigma_k\sigma_l\rangle \\
&\qquad -2\langle \sigma_i\sigma_m\rangle\langle \sigma_j\sigma_m\rangle(\langle \sigma_i\sigma_k\rangle - \langle \sigma_i\sigma_m\rangle\langle \sigma_m\sigma_k\rangle)(\langle \sigma_i\sigma_l\rangle - \langle \sigma_i\sigma_m\rangle\langle \sigma_m\sigma_l\rangle)
\end{aligned} \tag{1.3}$$

*Remark.* Above the critical temperature, the expression $\langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_j\rangle\langle \sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_k\rangle\langle \sigma_j\sigma_l\rangle - \langle \sigma_i\sigma_l\rangle\langle \sigma_j\sigma_k\rangle$ is the fourth Ursell function or truncated four-point function. Aizenman has recently shown[1] that

$$\begin{aligned}
&\langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_j\rangle\langle \sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_k\rangle\langle \sigma_j\sigma_l\rangle - \langle \sigma_i\sigma_l\rangle\langle \sigma_j\sigma_k\rangle \\
&\quad \geqslant -2\sum_m \langle \sigma_i\sigma_m\rangle\langle \sigma_j\sigma_m\rangle\langle \sigma_k\sigma_m\rangle\langle \sigma_l\sigma_m\rangle
\end{aligned}$$

and used this result to show that in greater than four dimensions hyperscaling does not hold for the Ising model, and the continuum limit of $\phi^4$ lattice models is a free field. Theorem 1 provides the complementary bound

$$\begin{aligned}
&\langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_j\rangle\langle \sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_k\rangle\langle \sigma_j\sigma_l\rangle - \langle \sigma_i\sigma_l\rangle\langle \sigma_j\sigma_k\rangle \\
&\quad \leqslant -2\langle \sigma_i\sigma_m\rangle\langle \sigma_j\sigma_m\rangle\langle \sigma_k\sigma_m\rangle\langle \sigma_l\sigma_m\rangle
\end{aligned} \tag{1.4}$$

The relation of (1.4) to Aizenman's inequality is similar to the relation of Griffiths inequality[2] $\langle \sigma_j\sigma_l\rangle \geqslant \langle \sigma_j\sigma_k\rangle\langle \sigma_k\sigma_l\rangle$ to Simon's inequality[3] $\langle \sigma_j\sigma_l\rangle \leqslant \sum_{k\in K}\langle \sigma_j\sigma_k\rangle\langle \sigma_k\sigma_l\rangle$, where $K$ is a set of sites that separates $j$ from $l$.

I am grateful to Alan Sokal for suggesting (1.4) to me.

Theorem 1 implies improvements of several known inequalities. Taking $m$ to be Griffiths "ghost" spin[2] we get the following.

**Corollary 1.** Consider an Ising model with Hamiltonian

$$-H = \sum_{1 \leqslant i < j \leqslant N} J_{ij}\sigma_i\sigma_j + \sum_{i=1}^{N} h_i\sigma_i$$

where $J_{ij} \geqslant 0$, $h_i \geqslant 0$. We have

$$\begin{aligned}\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle - \langle \sigma_i \sigma_j \rangle \langle \sigma_k \sigma_l \rangle - \langle \sigma_i \sigma_k \rangle \langle \sigma_j \sigma_l \rangle - \langle \sigma_i \sigma_l \rangle \langle \sigma_j \sigma_k \rangle \\ \leqslant -2\langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_k \rangle \langle \sigma_l \rangle - 2\langle \sigma_i ; \sigma_k \rangle^T \langle \sigma_j ; \sigma_k \rangle^T \langle \sigma_k \sigma_l \rangle \\ -2\langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_i ; \sigma_k \rangle^T \langle \sigma_i ; \sigma_l \rangle^T\end{aligned}$$

where by definition $\langle \sigma_i ; \sigma_k \rangle^T \equiv \langle \sigma_i \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_k \rangle$. Corollary 1 strengthens the Lebowitz inequality for the four-point function.[4–6]

If we take $m = l$ in (1.3) we get the following.

**Corollary 2.** Let $J_{ij} \geqslant 0$ in (1.1). Then

$$\begin{aligned}\langle \sigma_i \sigma_j \sigma_k \sigma_l \rangle - \langle \sigma_i \sigma_j \rangle \langle \sigma_k \sigma_l \rangle - \langle \sigma_i \sigma_k \rangle \langle \sigma_j \sigma_l \rangle - \langle \sigma_i \sigma_l \rangle \langle \sigma_j \sigma_k \rangle \\ \leqslant -2\langle \sigma_i \sigma_l \rangle \langle \sigma_j \sigma_l \rangle \langle \sigma_k \sigma_l \rangle \\ -2(\langle \sigma_i \sigma_k \rangle - \langle \sigma_i \sigma_l \rangle \langle \sigma_l \sigma_k \rangle)(\langle \sigma_j \sigma_k \rangle - \langle \sigma_j \sigma_l \rangle \langle \sigma_l \sigma_k \rangle) \langle \sigma_k \sigma_l \rangle \quad (1.5)\end{aligned}$$

This strengthens a result of GHS.[7]

If in (1.5) we now let $l$ be the "ghost" spin we have the following improvement of the GHS inequality.[7]

**Corollary 3.** Consider an Ising model with Hamiltonian

$$-H = \sum_{1 \leqslant i < j \leqslant N} J_{ij} \sigma_i \sigma_j + \sum_{i=1}^{N} h_i \sigma_i$$

where $J_{ij} \geqslant 0$, $h_i \geqslant 0$. We have

$$\langle \sigma_i ; \sigma_j ; \sigma_k \rangle^T \leqslant -2\langle \sigma_i ; \sigma_k \rangle^T \langle \sigma_j ; \sigma_k \rangle^T \langle \sigma_k \rangle \quad (1.6)$$

As usual $\langle \sigma_i ; \sigma_j ; \sigma_k \rangle^T \equiv \langle \sigma_i \sigma_j \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_j \sigma_k \rangle - \langle \sigma_j \rangle \langle \sigma_i \sigma_k \rangle - \langle \sigma_k \rangle \langle \sigma_j \sigma_i \rangle + 2\langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_k \rangle$.

We now investigate some of the consequences of Corollary 3. Consider an Ising model in a uniform positive external field $h$, and let $m_k(h)$ denote the magnetization at site $k$. If we sum (1.6) over $i$ and $j$ we have

$$\frac{\partial^2 m_k(h)}{\partial h^2} + 2\left[ \frac{\partial m_k(h)}{\partial h} \right]^2 m_k(h) \leqslant 0$$

By the "fluctuation-dissipation" relations[4] we have in the thermodynamic limit

$$\frac{\partial^2 m}{\partial h^2} + 2\left( \frac{\partial m}{\partial h} \right)^2 m \leqslant 0 \quad (1.7)$$

[4] For full justification of the fluctuation-dissipation relations, see for example, Sokal.[8] Other treatments are given by Fisher,[9] Sylvester,[10] and Lebowitz.[11]

(1.7) may be integrated twice to yield the following implicit bound on $m(h)$ in terms of $m_0 \equiv \lim_{h\to 0+} m(h)$ and $\chi_0 \equiv \lim_{h\to 0+} (\partial m/\partial h)(h)$.

**Theorem 2.** Let $m(h)$ denote the magnetization of an infinite volume Ising model in a uniform magnetic field $h > 0$. We have

$$\int_{m_0}^{m(h)} e^{y^2} dy \leqslant e^{m_0^2} \chi_0 h \tag{1.8}$$

*Proof.* Since $\partial m/\partial h$ is positive we can write (1.7) as

$$\frac{d}{dh}\left(\ln \frac{\partial m}{\partial h} + m^2\right) \leqslant 0 \quad \text{which implies} \quad \ln \frac{\partial m}{\partial h} + m^2 \leqslant \ln \chi_0 + m_0^2$$

Exponentiating and integrating from 0 to $h$ we get (1.7) ■

(1.7) also implies a critical exponent inequality. Writing $f(x) \sim x^\lambda$ to mean $\lim_{x\to 0+} [\log f(x)/\log x]$ exists and equals $\lambda$ we assume for $\beta > \beta_c$ ($T < T_c$) and $h = 0+$ (the plus state) that $M \sim (\beta - \beta_c)^\beta$

$$\begin{aligned} \frac{\partial m}{\partial h} &\equiv \chi \sim (\beta - \beta_c)^{-\gamma'} \\ \left|\frac{\partial^2 m}{\partial h^2}\right| &\sim (\beta - \beta_c)^{-\gamma' - \Delta_3'} \end{aligned} \tag{1.9}$$

(The exponent $\beta$ is not to be confused with the inverse temperature $\beta$.) See Stanley[12] or Sokal[8] for more complete lists of critical exponents.

(1.9) together with (1.7) imply immediately the following.

**Theorem 3.** Let $\beta$, $\gamma'$ and $\Delta_3'$ be defined as in (1.9). Then

$$\Delta_3' \geqslant \gamma' - \beta \tag{1.10}$$

*Remark.* (1.10) may also be written $\Delta_3' \geqslant \Delta_2' - 2\beta$. Numerical studies for the two-dimensional Ising model indicate $\Delta_3' \approx 15/8$, $\Delta_2' \approx 15/8$ while $\beta = 1/8$ is a rigorous result.[13] Hence in this case (1.10) reduces to $15/8 \geqslant 13/8$.

## 2. PROOF OF THEOREM 1

Without loss, we set $\beta = 1$. The proof is based on graphical methods. We refer the reader to Ref. 14 for notation and the proof of the following lemma.

**Lemma 1.** Let $J_{ij} \geqslant 0$ in (1.1). Then

$$(C(k) \not\ni 0 \mid \{j,k\},\{k,l\})(\phi)^2$$
$$\geqslant (C(k) \not\ni 0 \mid \{j,k\},\phi)(C(k) \not\ni 0 \mid \{k,l\},\phi)$$

The other ingredient we will need is the following result of GHS [7].

**Lemma 2.** Let $V_1$ and $V_2$ be sets of sites. Then

$$\sum_{\substack{\partial \mathbf{n}_1 = V_1 \\ \partial \mathbf{n}_2 = V_2}} w(\mathbf{n}_1)w(\mathbf{n}_2) = \sum_{\substack{\partial \mathbf{n}_1 = V_1 \Delta V_2 \\ \partial \mathbf{n}_2 = \phi}}' w(\mathbf{n}_1)w(\mathbf{n}_2)$$

where the primed summation has the restriction that $\mathbf{n}_1 + \mathbf{n}_2$ has a subgraph $\mathbf{s}$ with $\partial \mathbf{s} = V_2$. ($\Delta$ is the usual symmetric difference.)

We now turn to the proof. Consider

$$Z^4(\langle \sigma_i\sigma_j\sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_j\rangle\langle \sigma_k\sigma_l\rangle - \langle \sigma_i\sigma_k\rangle\langle \sigma_j\sigma_l\rangle - \langle \sigma_i\sigma_l\rangle\langle \sigma_j\sigma_k\rangle$$
$$+ 2\langle \sigma_i\sigma_m\rangle\langle \sigma_j\sigma_m\rangle\langle \sigma_k\sigma_m\rangle\langle \sigma_l\sigma_m\rangle)$$
$$= (\phi)^2((ijkl)(\phi) - (ij)(kl) - (ik)(jl) - (il)(jk))$$
$$+ 2(im)(jm)(km)(lm)$$
$$= (\phi)^2((ijkl)(\phi) - (C(i) \ni j \mid \{i,j,k,l\},\phi) - (C(i) \ni k \mid \{i,j,k,l\},\phi)$$
$$- (C(i) \ni l \mid \{i,j,k,l\},\phi))$$
$$+ 2(C(i) \ni m \mid \{i,j\},\phi)(C(k) \ni m \mid \{k,l\},\phi)$$

by Lemma 2

$$= -2(\phi)^2(C(i) \ni j,k,l \mid \{i,j,k,l\},\phi)$$
$$+ 2(C(i) \ni m \mid \{i,j\},\phi)(C(k) \ni m \mid \{k,l\},\phi) \qquad (2.1)$$

(2.1) follows from the observation that by flux conservation $C_{\mathbf{n}_1+\mathbf{n}_2}(i)$ contains either exactly one of the sites $j,k,l$ or all three of them. In the first case we have an exact cancellation and in the second we are left with $-2(\phi)^2(C(i) \ni j,k,l \mid \{i,j,k,l\},\phi)$. Using Lemma 2, (2.1) may be written

$$-2(\phi)^2(C(i) \ni k \mid \{i,j\},\{k,l\})$$
$$+ 2(C(i) \ni m \mid \{i,j\},\phi)(C(k) \ni m \mid \{k,l\},\phi)$$
$$= 2(\phi)^2(C(i) \not\ni k \mid \{i,j\},\{k,l\})$$
$$- 2(C(i) \not\ni m \mid \{i,j\},\phi)(kl)(\phi)$$
$$- 2(C(i) \ni m \mid \{i,j\},\phi)(C(k) \not\ni m \mid \{k,l\},\phi)$$

by adding and subtracting $2(\phi)^2(ij)(kl)$

$$
\begin{aligned}
&= -2(\phi)(C(i) \not\ni m, C(i) \ni k \text{ or } l \mid \{i,j\},\phi)(kl) \\
&\quad - 2(\phi)(C(i) \not\ni m,k,l \mid \{i,j\},\phi)(kl) \\
&\quad + 2(\phi)^2(C(i) \not\ni m,k \mid \{i,j\},\{k,l\}) \\
&\quad - 2(im)(jm)(C(k) \not\ni m, C(k) \ni i \text{ or } j \mid \{k,l\},\phi) \\
&\quad - 2(im)(jm)(C(k) \not\ni m,i,j \mid \{k,l\},\phi) \\
&\quad + 2(\phi)^2(C(i) \not\ni k, C(i) \ni m \mid \{i,j\},\{k,l\})
\end{aligned}
$$

where we have again used Lemma 2

$$
\begin{aligned}
&= -2(\phi)(C(i) \not\ni m, C(i) \ni k \text{ or } l \mid \{i,j\},\phi)(kl) \\
&\quad - 2(\phi)\sum_A{}' \left(C^b(i) \equiv A \mid \{i,j\},\phi\right)_A (\phi)^2_{\bar{A}^c}(\phi)(\langle\sigma_k\sigma_l\rangle - \langle\sigma_k\sigma_l\rangle_{\bar{A}^c}) \\
&\quad - 2(im)(jm)(C(k) \not\ni m, C(k) \ni i \text{ or } j \mid \{k,l\},\phi) \\
&\quad - 2\sum_B{}'' \left(C^b(k) \equiv B \mid \{k,l\},\phi\right)_B (\phi)^2_{\bar{B}^c}(\phi)^2 \\
&\quad \times (\langle\sigma_i\sigma_m\rangle\langle\sigma_j\sigma_m\rangle - \langle\sigma_i\sigma_m\rangle_{\bar{B}^c}\langle\sigma_j\sigma_m\rangle_{\bar{B}^c})
\end{aligned} \tag{2.2}
$$

where the single-primed summation is over connected sets of bonds $A$ sucl that $i$ and $j$ belong to at least one of the bonds and $m$, $k$, and $l$ belong t( none of them. The double-primed summation is over connected sets o bonds $B$ such that $k$ and $l$ belong to at least one of the bonds and $m$, $i$, an( $j$ belong to none of them. We have used Lemma 2 in the last term.

The GKS inequalities[2,15] imply that $\langle\sigma_k\sigma_l\rangle - \langle\sigma_k\sigma_l\rangle_{\bar{A}^c} \geqslant 0$ and henc (2.2) is not larger than

$$
\begin{aligned}
&-2(\phi)(C(i) \not\ni m, C(i) \ni k \text{ or } l \mid \{i,j\},\phi)(kl) \\
&\quad - 2(im)(jm)(C(k) \not\ni m, C(k) \ni i \text{ or } j \mid \{k,l\},\phi) \\
&\leqslant -2(\phi)(C(i) \not\ni m \mid \{i,k\},\{k,j\})(kl) \\
&\quad - 2(im)(jm)(C(k) \not\ni m \mid \{k,i\},\{i,l\})
\end{aligned}
$$

using Lemma 2

$$
\begin{aligned}
&\leqslant \frac{-2(kl)}{(\phi)}(C(i) \not\ni m \mid \{i,k\},\phi)(C(k) \not\ni m \mid \{k,j\},\phi) \\
&\quad \frac{-2(im)(jm)}{(\phi)^2}(C(k) \not\ni m \mid \{k,i\},\phi)(C(i) \not\ni m \mid \{i,l\},\phi)
\end{aligned}
$$

by Lemma 1.

Since $(C(i) \not\ni m \mid \{i,k\}, \phi) = (ik)(\phi) - (im)(km)$ by Lemma 2, Theorem 1 follows by dividing by $Z^4$.

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## REFERENCES

1. M. Aizenman, *Phys. Rev. Lett.* **47**:1 (1981), and Geometric analysis of $\phi^4$ fields and Ising models, to appear in *Commun. Math. Phys.*
2. R. Griffiths, *J. Math. Phys.* **8**:484 (1967).
3. B. Simon, *Commun. Math. Phys.* **77**:111 (1980).
4. J. L. Lebowitz, *Commun. Math. Phys.* **35**:87 (1974).
5. G. S. Sylvester, *J. Stat. Phys.* **15**:327 (1976).
6. R. S. Ellis, J. L. Monroe, and C. M. Newman, *Commun. Math. Phys.* **46**:167 (1976).
7. R. Griffiths, C. Hurst, and S. Sherman, *J. Math. Phys.* **11**:790 (1970).
8. A. D. Sokal, *J. Stat. Phys.* **25**:25 (1981).
9. M. E. Fisher, *Phys. Rev.* **180**:594 (1969).
10. G. S. Sylvester, Continuous-spin Ising ferromagnets, MIT thesis (1976), Chap. V.
11. J. L. Lebowitz, *Commun. Math. Phys.* **28**:313 (1972).
12. H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, England, 1971).
13. A. Martin-Löf, *Commun. Math. Phys.* **24**:253 (1972).
14. R. Graham, *J. Stat. Phys.* **29**:177 (1982).
15. D. G. Kelly and S. Sherman, *J. Math. Phys.* **9**:466 (1968).